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Analysis of Non-Relativistic Energy, Wave Function, and Black Hole's Application with Minimal Length

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Abstrak

Persamaan Schrödinger untuk potensial Eckart termodifikasi q-deformasi diselidiki dengan adanya formalisme panjang minimal menggunakan perkiraan fungsi gelombang baru dan potensi q-deformed. Itu direduksi menjadi persamaan Schrödinger dengan potensi Poschl-Teller yang dimodifikasi. Persamaan ini diselesaikan dengan menggunakan Metode Iterasi Asimtotik (AIM) untuk mendapatkan persamaan nilai eigen energi dan fungsi gelombang. Fungsi gelombang digunakan untuk menentukan entropi Rényi sistem kuantum. Kemudian, entropi Rényi digunakan untuk menentukan parameter energi massa, suhu dan kapasitas panas lubang hitam untuk beberapa molekul diatomik. Spektrum energi menunjukkan bahwa peningkatan bilangan kuantum radial dan sudut (*n, L*), lebar potensial (*γ*), parameter panjang minimal (α_{ML}), dan massa molekul (*m*), menyebabkan penurunan nilai eigen energi. Bilangan kuantum radial dan parameter paling berpengaruh terhadap fungsi gelombang, jumlah gelombang, dan panjang gelombang. Parameter lebar potensial, bilangan kuantum radial, dan entropi paling berpengaruh terhadap entropi Rényi, parameter energi massa, suhu dan kapasitas panas lubang hitam Schwarzschild. Peningkatan *n* dan *γ* menyebabkan penurunan R_d, M, T_R, dan C_R tetapi peningkatan *d* menyebabkan peningkatan R_d, M, T_R, dan C_R. Besaran-besaran ini adalah kunci untuk menganalisis lebih lanjut fitur karakteristik dari suatu sistem atau partikel.

Kata Kunci: persamaan Schrödinger; panjang minimal; Metode Iterasi Asimtotik (AIM), entropi Renyi; sifat lubang hitam.

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Abstract

The Schrödinger equation for q-deformed modified Eckart potential was investigated in the presence of minimal length formalism using the approximate new wave function and q-deformed potential. It was reduced to the Schrödinger equation with modified Poschl-Teller potential. This equation was solved using the Asymptotic Iteration Method (AIM) to get the energy eigenvalues equation and wave functions. The wave function was used to determine the Rényi entropy of a quantum system. Then, the Rényi entropy was used to determine the mass-energy parameter, temperature and heat capacity of the black hole for some diatomic molecules. The energy spectra showed that the increase of radial and angular quantum number *(*n, L*),* potential width *(*γ*),* minimal length parameter *(*α_{ML}*),* and the molecule mass *(*m*),* caused the decrease of energy eigenvalues. The radial quantum number and the *γ* parameter had the most effect on the wave functions, the number of waves, and the wavelength. The potential width, radial quantum number, and entropy parameter had the most effect on Rényi entropy, mass-energy parameter, temperature, and heat capacity of Schwarzschild black hole. The increase of *n* and *γ* caused the decrease of R_d, M, T_R, and C_R, but the increase of *d* caused the increase of R_d, M, T_R, and C_R. These properties are the key to further analyzing the characteristic features of a system or particle.

Keywords*: Schrödinger equation; minimal length; Asymptotic Iteration Method (AIM), Rényi entropy; black hole properties.*

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INTRODUCTION

An analytical solution of the radial Schrödinger equation is very important in quantum mechanics because the wave functions and energy eigenvalues contain all the important information needed to describe quantum systems (Bayrak et al., 2007). Researchers have been conducting this research for years to investigate the solution of wave and energy equations for various potentials. Some potentials are solved exactly (Hassanabadi et al., 2013; Ikot et al., 2010) while others can only be solved by using a different form of approach (Berkedemir and Han, 2005; Okon et al., 2017; Pekeris, 1934). Several different methods also have been used to solve the Schrödinger equation for wave functions and energy with different potentials, such as the factorization method (Dong, 2007), supersymmetry of quantum mechanics (SUSYQM) (Ahmadov et al., 2017; Ahmadov et al., 2018), Asymptotic Iteration Method (AIM) (Falaye, 2012; Bayrak and Boztosun, 2007), Nikiforov–Uvarov method (NU) (Berkdemir and Han, 2005; Ikot et al., 2013).

The minimal length is the canonical commutation relationship between position and momentum operators by adding a small correction of a constant quantity in the Heisenberg uncertainty principle. The minimal length is also known as the Generalized Uncertainty Principle (GUP). The minimal length parameter is an additional correction due to the presence of quantum gravity on the system such as string theory and a black hole in the form of a constant parameter . The minimal length parameter has an interval of . The minimal length equation returns to the Heisenberg uncertainty equation if the energy of the particle system is smaller than the Planck mass and is close to zero (Alimohammadi and Hassanabadi, 2017; Haouat, 2014).

The deformation of q in a group of quantum systems has been developed gradually in mathematics, chemistry, and physics. In physics studied in detail are nuclear physics and high energy (Zhand and Osland, 2001; Lavagno et al., 2006; Dianawati et al., 2018), black hole (Lerda, 1992), cosmic strings (Strominger, 1993), the q-deformed harmonic oscillator (Macfarlane, 1989), expression of oscillatory-rotational spectra of multi-atomic molecules (Sari et al., 2015), and q-deformed exponential or hyperbolic potential (Bonastos et al., 1997; Johal and Gupta, 1998).

The Bohr Mottelson equation in minimal length formalism for the q-deformed modified Eckart potential has been solved by the hypergeometry method (Suparmi et al., 2019) but the research has not been used to determine Rényi entropy and thermodynamical properties of a black hole. Therefore, in this paper, the Schrödinger equation in minimal length formalism for q-deformed modified Eckart potential will be solved using Asymptotic Iteration Method (AIM) to obtain the energy spectra and its corresponding wave function which will be used to determine Rényi entropy and thermodynamical properties of a black hole (Czinner and Iguchi, 2016; Biro and Czinner, 2013). The radial part of Schrödinger equation in minimal length formalism for q-deformed modified Eckart potential is solved using an approximate solution by introducing a new wave function (Chabab et al., 2016) such that it is reduced to a usual Schrödinger like equation that is solvable by AIM.

In this work, we examine the energy eigenvalue and eigenfunction of a system influenced by q-deformed modified Eckart potential and also with the presence of minimal length which has not been investigated before. Furthermore, we also apply the energy eigenvalue and eigenfunction equations to the diatomic molecules (CO, NO, O2, I2) computation of Renyi entropy and the thermodynamics properties of the Schwarzschild black hole. The thermodynamics properties are an important parameter to analyze the characteristic features of a system or particle.

We have simulated and analyzed the mass-energy parameter, the temperature, and also the heat capacity of the Schwarzschild black hole.

This paper is structured as follows. In Section 2, the approximate solution of the Schrödinger equation with minimal length effect is briefly described. The q-deformed modified Eckart potential is described in Section 3. The q-deformed hyperbolic potential properties potential is described in Section 4 The Asymptotic Iteration Method (AIM) is briefly reviewed in Section 5. The Rényi entropy and its application to a black hole are reviewed in Section 6. The result and discussions about the energy spectrum, wave function, Rényi entropy, and application black hole are presented in Section 7. Finally, in Section 8, the conclusion is presented.

THE APPROXIMATE SOLUTION OF THE SCHRÖDINGER EQUATION WITH MINIMAL LENGTH EFFECT

The particle dynamic in quantum mechanics corresponds to the position and momentum of particles. The study of commutation relations between position and momentum operators is explained using the Heisenberg uncertainty principle (Sprenger *et al.,* 2012; Garay, 1995), given by

$$\left[\hat{x}, \hat{p}\right] \geq i\hbar \qquad (1)$$

where $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, i is an imaginary number, and $\hbar = \frac{h}{2}$ with h is Planck constant. This important idea can be explained by noncommutative geometry (Bonatsos *et al.,* 2006) in the quantum gravity (Ciftci *et al.,* 2003; Pratiwi *et al.,* 2017; Pramono *et al.,* 2016) and the string theory context (Hulthe and Sugawara, 1957; Varshni, 1990).

The presence of quantum gravity on quantum mechanics can find the existence of a minimal observable distance on the order of the Planck length. Therefore, the Heisenberg uncertainty principle gets additional correction due to the presence of quantum gravity, which is well known as the Generalized Uncertainty Principle (GUP) (Alimohammadi and Hassanabadi, 2017; Chabab *et al.,* 2016), given by

$$\left[X, P\right] = i\hbar\left(1 + \alpha_{ML} p^2\right) \qquad (2)$$

where α_{ML} is a minimal length parameter which is in the range of $0 \leq \alpha_{ML} \leq 1$, P is the quantity of momentum in high energy and p is the quantity of momentum in low energy. In GUP parameters α_{ML} should be calculated from fundamental theory. When energy is smaller than the Planck mass scale, α_{ML} approaches zero and we can restore to the Heisenberg uncertainty principle (Alimohammadi and Hassanabadi, 2017; Chabab *et al.,* 2016)

The uncertainty of particle position is commutation relations between position and momentum operators which are expressed in the Heisenberg uncertainty principle. From Equation (2) we have (Alimohammadi and Hassanabadi, 2017; Chabab *et al.,* 2016),

$$\hat{X} = \hat{x} \qquad (3)$$

$$\hat{P}_i = \left(1 + \alpha_{ML} \hat{p}^2\right) \hat{p}_i \qquad (4)$$

where $\hat{P}_i$ and $\hat{p}_i$ are related momentum operators to high and low energies, respectively. The momentum operator $\hat{p}$ is expressed by $p = -i\hbar\nabla$.

The time-independent Schrödinger equation with the presence of minimal length is given as

$$\frac{\left(1+\alpha_{ML}\,p_i^{\,2}\right)^2\,p_i^{\,2}}{2m}\psi+V\psi=E\psi \qquad (5)$$

here, α_{ML} is a very small quantity, therefore $\alpha_{ML}^2\approx 0$ then Equation (5) becomes

$$\left(\frac{-\hbar^2\Delta}{2m}+\frac{2\alpha_{ML}\hbar^4\Delta^2}{2m}\right)\psi+V\psi=E\psi \quad (6)$$

The new wave function that has to be introduced to get rid of the Δ^2 term in Equation (6) is given as (Chabab *et al.,* 2016)

$$\psi\left(r,\theta,\varphi\right)=\left(1+2\alpha_{ML}\hbar^2\Delta\right)\phi\left(r,\theta,\varphi\right) \quad (7)$$

Equation (7) is modified to eliminate quadratic Laplacian such that we get Equation (8) in reference (Chabab *et al.,* 2016). By substituting Equation (7) into Equation (6), and due to $\alpha_{ML}^2\approx 0$, then

$$-\frac{\hbar^2\Delta\phi}{2m}+2\alpha_{ML}\hbar^2V\Delta\phi-2\alpha_{ML}\hbar^2E\Delta\phi=0 \quad (8)$$

By multiplying Equation (8) with $-\frac{2m}{\hbar^2}$, we get

$$\Delta\phi=\frac{\frac{2m}{\hbar^2}\left(V-E\right)\phi}{1-4m\alpha_{ML}\left(V-E\right)} \qquad (9)$$

By using simple mathematical manipulation, Equation (9) becomes

$$\Delta\phi-\frac{-\frac{2m}{\hbar^2}E}{\left(1+4m\alpha_{ML}E\right)}\frac{\left(1-\frac{V}{E}\right)\phi}{\left(1-\frac{4m\alpha_{ML}V}{1+4m\alpha_{ML}E}\right)}=0 \quad (10)$$

By setting the new parameters in Equation (10)

$$\begin{aligned}&\frac{-\frac{2m}{\hbar}E}{\left(1+4m\alpha_{ML}E\right)}=\frac{\upsilon}{\mu};\\&\frac{-4m\alpha_{ML}}{1+4m\alpha_{ML}E}=4\tau;\\&E=-\frac{1}{\rho}\end{aligned} \qquad (11)$$

Then by using simple mathematical manipulation from Equations (10) and (11) we obtain

$$\Delta\phi-\frac{\upsilon}{\mu}\left\{1+\frac{\left(\rho-4\tau\right)V}{\left(1+4\tau V\right)}\right\}\phi=0 \qquad (12)$$

The Schrödinger Equation in the presence of minimal length expressed in equation (12) given as

$$\left\{\begin{array}{l}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}r^2\dfrac{\partial}{\partial r}-\dfrac{\upsilon}{\mu}\left\{1+\dfrac{(\rho-4\tau)V}{(1+4\tau V)}\right\}\\ +\dfrac{1}{r^2}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\sin\theta\dfrac{\partial}{\partial\theta}+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\varphi^2}\right]\end{array}\right\}\phi=0 \quad (13)$$

By setting $\phi(r,\theta,\varphi)=\dfrac{R(r)}{r}\Theta(\theta)\Phi(\varphi)$ then the polar part of Schrödinger Equation is given as,

$$-\left[\begin{array}{l}\dfrac{1}{\Theta(\theta)}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\sin\theta\dfrac{\partial\Theta(\theta)}{\partial\theta}\\ +\dfrac{1}{\Phi(\varphi)}\dfrac{1}{\sin^2\theta}\dfrac{\partial^2\Phi(\varphi)}{\partial\varphi^2}\end{array}\right]=L(L+1) \quad (14)$$

With *L* is the angular quantum number, and the radial part of the Schrödinger Equation is given as

$$\left(\begin{array}{l}\dfrac{d^2\chi(r)}{dr^2}\\ -\dfrac{\upsilon}{\mu}\left\{1+\dfrac{(\rho-4\tau)V(r)}{(1+4\tau V(r))}\right\}\chi(r)\end{array}\right)=\frac{L(L+1)}{r^2}\chi(r) \quad (15)$$

Q-DEFORMED MODIFIED ECKART POTENTIAL

The Eckart potential is an asymmetric function. The interesting results concerning the potential give inspiration not only to explore another similar potential but also to study the thermal reaction of Formaldehyde (Ikhdair and Falaye, 2014). This potential is exactly solvable or quasi exactly solvable and their bound state solutions have been reported (Suparmi *et al.,* 2013). The general form of Eckart potential is given as

$$V(r)=\frac{1}{a^2}\left(V_O\frac{e^{-\frac{r}{a}}}{(1-e^{-\frac{r}{a}})^2}-V_1\frac{1+e^{-\frac{r}{a}}}{1-e^{-\frac{r}{a}}}\right) \quad (16)$$

with V_O and V_1 describe the depth of the potential well and are positives, $V_1>V_O$, *a* is a positive parameter that controls the width of the potential well $0<\dfrac{r}{a}<\infty$. In this study, we apply a special case of Eckart potential in Equation (16) by setting, $V_1=0$, $\gamma=\dfrac{1}{a}$, and applying spatial deformation to the potential such that we get q-deformed modified Eckart potential, with the deformation parameter q.

$$V=\frac{V_0\gamma^2e^{-2\gamma r}}{\left(1-V_0\gamma^2\tau e^{-2\gamma r}\right)^2}=\frac{V_0\gamma^2}{\left(e^{\gamma r}-qe^{-\gamma r}\right)^2} \quad (17)$$

Q-DEFORMED HYPERBOLIC POTENTIAL PROPERTIES

The q-deformed potential is a special kind of f-deformed potential with only one deformed parameter q (Darareh and Harouni, 2010). The quantum deformation of hyperbolic potential was introduced by Arai (1991). It has been investigated by some authors, Eshghi (2012), Akpan *et al* (2012), Egrifes et al (1999), Dutra

(2005), Suparmi *et al*. (2013). Dutra has reinterpreted the idea of quantum deformation as a kind of parameter scaling symmetry of the model, so the q-deformed potential is not a new class of potential. By a convenient translation of spatial variables, one can transform the deformed potentials into the form of non-deformed potentials or vice-versa. In analogy to the translation of spatial variable for the hyperbolic function which is introduced by Dutra (2005), by setting the translation of spatial variable for hyperbolic function as

$$r \to r + \frac{\ln\sqrt{q}}{\gamma} \qquad (18)$$

then Equation (18) is applied into q deformed hyperbolic potential (Arai, 1991) we obtain

$$\begin{aligned}\sinh_q \gamma r &= \frac{e^{\gamma\left(r+\frac{\ln\sqrt{q}}{\gamma}\right)} - qe^{-\gamma\left(r+\frac{\ln\sqrt{q}}{\gamma}\right)}}{2} \\ &= \frac{e^{\gamma r}e^{\ln\sqrt{q}} - qe^{-\gamma r}e^{-\ln\sqrt{q}}}{2} = \sqrt{q}\sinh\gamma r \qquad (19) \\ \cosh_q \gamma r &\to \sqrt{q}\cosh\gamma r\end{aligned}$$

ASYMPTOTIC ITERATION METHOD (AIM)

Asymptotic Iteration Method (AIM) is one method to obtain the exact solution of the second-order differential equation is written as (Falaye, 2012; Bayrak and Boztosun, 2012; Elviyanti *et al., 2018*)

$$y_n''(x) - \lambda_0(x)y_n'(x) - s_0(x)y_n(x) = 0 \qquad (20)$$

where $\lambda_0(x) = 0,$ and $s_0(x)$ is the coefficient of the differential equation and *n* denotes the quantum number. Equation (20) is differentiated to *x,* we obtain Equation (21),

$$y_n'''(x) - \lambda_1(x)y_n'(x) - s_1(x)y_n(x) = 0 \qquad (21)$$

where $\lambda_1(x) = \lambda_0' + \lambda_0^2 + s_0$ and $s_1(x) = s_0' + s_0\lambda_0$. The second derivative of Equation (20) is

$$y_n''''(x) - \lambda_2(x)y_n'(x) - s_2(x)y_n(x) = 0 \qquad (22)$$

where $\lambda_2(x) = \lambda_1' + \lambda_1\lambda_0 + s_1$, $s_2(x) = s_1' + s_0\lambda_1$, and so on until the *i*-th derivative of Equation (20) is given as

$$y_n^{\,i}(x) - \lambda_{i-2}(x)y_n'(x) - s_{i-2}(x)y_n(x) = 0 \qquad (23)$$

with

$$\lambda_i(x) = \lambda_{i-1}' + \lambda_{i-1}\lambda_0 + s_{i-1}; s_i(x) = s_{i-1}' + \lambda_{i-1} \qquad (24)$$

The eigenvalue of second-order differential equation which is expressed in Equation (26) that is solved using AIM is obtained from the quantization condition which is given by,

$$\begin{aligned}&\lambda_i(x)s_{i-1}(x) - \lambda_{i-1}(x)s_i(x) = 0 \\ &= \Delta_i, i = 1,2,3\ldots.\end{aligned} \qquad (25)$$

By rewriting Equation (20) as

$$y''(x)=\left\{\begin{array}{c}2\left(\dfrac{tx^{N+1}}{1-bx^{N+2}}-\dfrac{c+1}{x}\right)y'(x)\\ -\dfrac{Wx^{N}}{1-bx^{N+2}}y(x)\end{array}\right\} \quad (26)$$

then we get the solution of Equation (26) in form of the hypergeometric term, given as,

$$y_n(x)=\left\{\begin{array}{c}((-1)^n C'(N+2)^n(\delta)_n\\ {}_2F_1\left(-n,s+n,\delta,bx^{N+2}\right)\end{array}\right\} \quad (27)$$

$$\delta=\frac{2c+N+3}{N+2} \text{ and } s=\frac{(2c+1)b+2t}{(N+2)b} \quad (28)$$

C' is normalization constant.

RÉNYI ENTROPY AND ITS APPLICATION TO BLACK HOLE

1. Rényi entropy

The Rényi entropy is the generalization of Shannon entropy $d\to 1$ that measures the uncertainty of particle location in space. The Rényi entropy reduces to Shannon entropy. The Rényi was introduced by Rényi (1960) as a generalization of the Shannon entropy which depends on a parameter *d.* The Rényi entropy is defined as (Rényi, 1960; Yahya *et al., 2015*)

$$R_d(\xi)=S_R=\frac{1}{1-d}\log 4\pi\int_0^\infty \xi(r)^d dr \quad (29)$$

2. Schwarzschild black hole

In the classical approach, the Schwarzschild black hole appears to be thermodynamically unstable in canonical treatment due to the frequent negativity of the heat capacity of the black hole. This was reported by Czinner and Iguchi (2016) as a conclusion from Hessian analysis. Recently, thermodynamics properties of Schwarzschild black hole that have been studied include the work of Biro and Czinner (2013), Czinner and Iguchi (2016). Here, the usual Rényi entropy (S_R) is used in quantum computation to compute the mass-energy parameter, that is used to determine the temperature (T_R), and heat capacity (C_R) of the black hole in terms of the work of Czinner and Iguchi (2016) as follows

$$S_R=\frac{1}{\lambda}\ln\left[1+\lambda S_{BH}\right] \quad (30)$$

and for the Schwarzschild solution, it results (Czinner and Iguchi, 2016)

$$S_R=\frac{1}{\lambda}\ln\left[1+4\pi\lambda M^2\right] \quad (31)$$

$$T_R=\frac{1}{8\pi M}+\frac{\lambda M}{2}=\frac{1+4\pi\lambda M^2}{8\pi M} \quad (32)$$

$$C_R=\frac{8\pi M^2}{4\pi\lambda M^2-1} \quad (33)$$

where S_{BH} is Bekenstein-Hawking entropy, with $\lambda=1-d$ and M is the mass-energy parameter of the black hole.

RESULTS AND DISCUSSION

Equation (15) cannot be solved exactly unless we use the approximation of centrifugal term, $1/r^2$, which is given as (Naderi and Hassanabadi, 2016)

$$\frac{1}{r^2} = \frac{\gamma^2}{\sinh^2(\gamma r)} \qquad (34)$$

The visualization of that approximation in Equation (34) is shown in Figure 1.

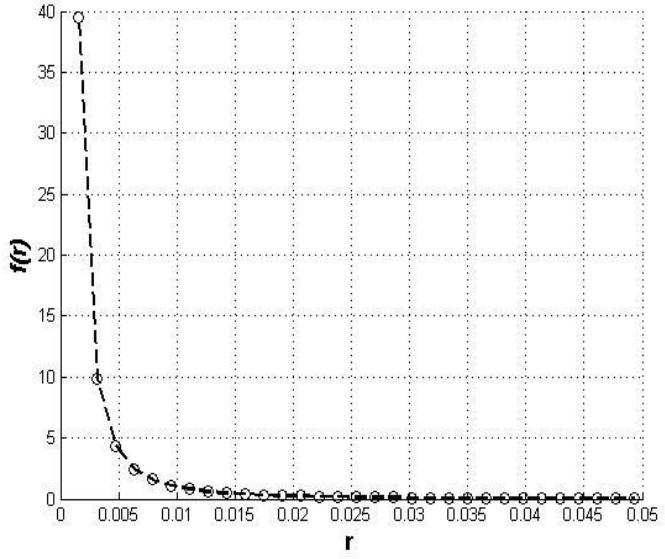

Figure 1. The visualization of $f(r)=\frac{1}{r^2}$ and $f(r)=\frac{r^2}{\sin^2(\gamma r)}$ for $\gamma = 0.1$, the dashed line (- - -) for $1/r^2$ and the circle line (o-o-o-o) for $\frac{\gamma^2}{\sinh^2(\gamma r)}$.

It is seen that the two lines overlap with each other, then the centrifugal term $1/r^2$ is matched to be approximated by $\frac{\gamma^2}{\sinh^2(\gamma r)}$. The exact solution of Equation (15) can be obtained by inserting Equations (34), (17) into Equation (15), so we get

$$\frac{\partial^2\chi}{\partial r^2} - \frac{\upsilon}{\mu}\left\{1+\frac{(\rho-4\tau)\frac{V_0\gamma^2 e^{-2\gamma r}}{\left(1-qe^{-2\gamma r}\right)^2}}{\left(1+4\tau\frac{V_0\gamma^2 e^{-2\gamma r}}{\left(1-qe^{-2\gamma r}\right)^2}\right)}\right\}\chi = \frac{L(L+1)}{\sinh^2\gamma r}\chi \qquad (35)$$

By inserting equation (19) into Equation (35) and then it is simplified by simple mathematical manipulation we get

$$\frac{\partial^2\chi}{\partial r^2} - \frac{L(L+1)\gamma^2}{\sinh^2\gamma r}\chi - \frac{\upsilon}{\mu}\left\{\frac{(\rho-4\tau)V_0\gamma^2}{4q\cosh^2\gamma r}\right\}\chi = \frac{\upsilon}{\mu}\chi \qquad (36)$$

By setting the new parameters in Equation (36)

$$L(L+1)=\kappa(\kappa-1);\ \frac{\upsilon}{\mu}=E';\ \tau V_0\gamma^2=q;$$
$$\frac{\upsilon(\rho-4\tau)V_0}{4q\mu}=\frac{\upsilon(\rho-4\tau)}{4\tau\mu\gamma^2}=-\sigma(\sigma+1) \quad (37)$$

and by inserting Equation (36)-(37) into Equation (35), we get

$$\frac{\partial^2\chi}{\partial r^2}-\left[\frac{\gamma^2\kappa(\kappa-1)}{\sinh^2\gamma r}-\frac{\gamma^2\sigma(\sigma+1)}{\cosh^2\gamma r}\right]\chi=E'\chi \quad (38)$$

Equation (38) is solved using the *Asymptotic Iteration Method* (AIM) by setting new variables as

$$\cosh^2(\gamma r)=z \quad (39)$$

$$\frac{\partial}{\partial r}=\frac{\partial z}{\partial r}\frac{\partial}{\partial z}=2\gamma\sqrt{z(z-1)}\frac{\partial}{\partial z};$$
$$\frac{\partial^2}{\partial r^2}=4\gamma^2 z(z-1)\frac{\partial^2}{\partial z^2}+2\gamma^2(2z-1)\frac{\partial}{\partial z} \quad (40)$$

By applying Equations (39)-(40) in Equation (38), we have

$$\left(\begin{array}{l} z(1-z)\dfrac{\partial^2\chi}{\partial z^2}+(\dfrac{1}{2}-z)\dfrac{\partial\chi}{\partial z} \\ +\left(\dfrac{k^2}{4}-\dfrac{\kappa(\kappa-1)}{4(1-z)}-\dfrac{\sigma(\sigma+1)}{4z}\right)\chi \end{array}\right)=0 \quad (41)$$

with $k^2=\dfrac{E'}{\gamma^2}$.

By setting the new wave function in equation (41) as

$$\chi(r)=\chi(z)=z^{\alpha}(1-z)^{\beta}\ f(z) \quad (42)$$

with

$$-\frac{\sigma}{2}=\alpha;\frac{\kappa}{2}=\beta \quad (43)$$

then Equation (41) becomes

$$z(1-z)\frac{d^2f}{dz^2}+\left(\begin{array}{l}\left(2\alpha+\dfrac{1}{2}\right) \\ -(2\alpha+2\beta+1)z\end{array}\right)\frac{df}{dz}$$
$$+\{\frac{k^2}{4}-(\alpha+\beta)^2\}f=0 \quad (44)$$

Equation (43) is reduced to the AIM-type differential equation that is similar to Equation (20) which is given as

$$\frac{d^2f}{dz^2}+2\left(\frac{\left(\alpha+\frac{1}{4}\right)}{z(1-z)}-\frac{\left(\alpha+\beta+\frac{1}{2}\right)}{1-z}\right)\frac{df}{dz}$$
$$+\left\{\frac{\frac{k^2}{4}-(\alpha+\beta)^2}{z(1-z)}\right\}f=0 \quad (45)$$

By comparing Equations (20) and (44), we have

$$\lambda_o = \frac{\left(2\alpha+\frac{1}{2}\right)-(2\alpha+2\beta+1)z}{z(1-z)} \qquad s_O = \frac{\frac{k^2}{4}-(\alpha+\beta)^2}{z(1-z)} \tag{46}$$

By using Matlab programming and using Equations (23)-(25) and (45)-(46) we get

$$\Delta_o \to \frac{k^2}{4} = (\alpha+\beta)^2;$$
$$\Delta_1 \to \frac{k^2}{4} = (\alpha+\beta+1)^2;$$
$$\Delta_2 \to \frac{k^2}{4} = (\alpha+\beta+2)^2; \tag{47}$$
$$\Delta_i \to \frac{k^2}{4} = (\alpha+\beta+i)^2$$

By generalizing Equation (47), we have the general formulation for spectrum energy as,

$$\frac{k^2}{4} = (\alpha+\beta+n)^2 \tag{48}$$

By inserting Equations (37) and (43) into Equation (48) we have

$$\frac{\upsilon}{\mu\gamma^2} = \frac{E'}{\gamma^2} = k^2 = (\sigma-\kappa-2n)^2 \tag{49}$$

From the combination of Equations (11) and (49) and by setting $E = E_n$, we get the energy spectrum equation given as

$$-E_n = \frac{\gamma^2(\sigma-\kappa-2n)^2}{\frac{2m}{\hbar^2}\left\{2\hbar^2\alpha_{ML}\gamma^2(\sigma-\kappa-2n)^2+1\right\}} \tag{50}$$

with

$$\sigma+\frac{1}{2} = \sqrt{\frac{1}{2\hbar^2\alpha_{ML}\left(1+4m\alpha_{ML}E_n\right)}}; \qquad \kappa = L+1 \tag{51}$$

Equation (51) is the energy spectrum equation of the quantum system for the q-deformed modified Eckart potential in the minimal length formalism. Table 1 shows the reduced mass of CO, NO, O_2, and I_2 molecules. The numerical calculation of the energy spectra of some diatomic molecules that are influenced by q-deformed modified Eckart potential, for various values of quantum number (*n, L*), potential width, and minimal length parameter (γ , α_{ML}), are shown in Figures.

Table 1**.** *The characteristic of spectroscopic and reduces mass for diatomic molecules in the ground state*

Parameter	**CO**	**NO**	**O_2**	**I_2**
m (amu)	6.860586000	7.468441000	7.997457504	63.45223502

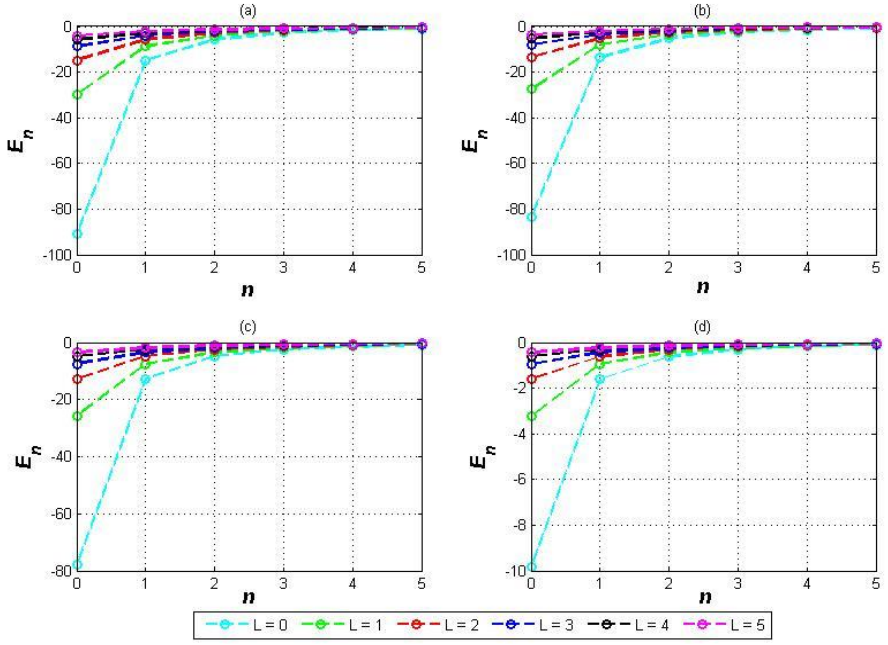

Figure 2. Spectrum Energy of q-deformed modified Eckart potential (E_n) in eV for the various n and L quantum numbers, and constant potential and minimal length parameter, γ=0.1 and α_{ML}=0.1 for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

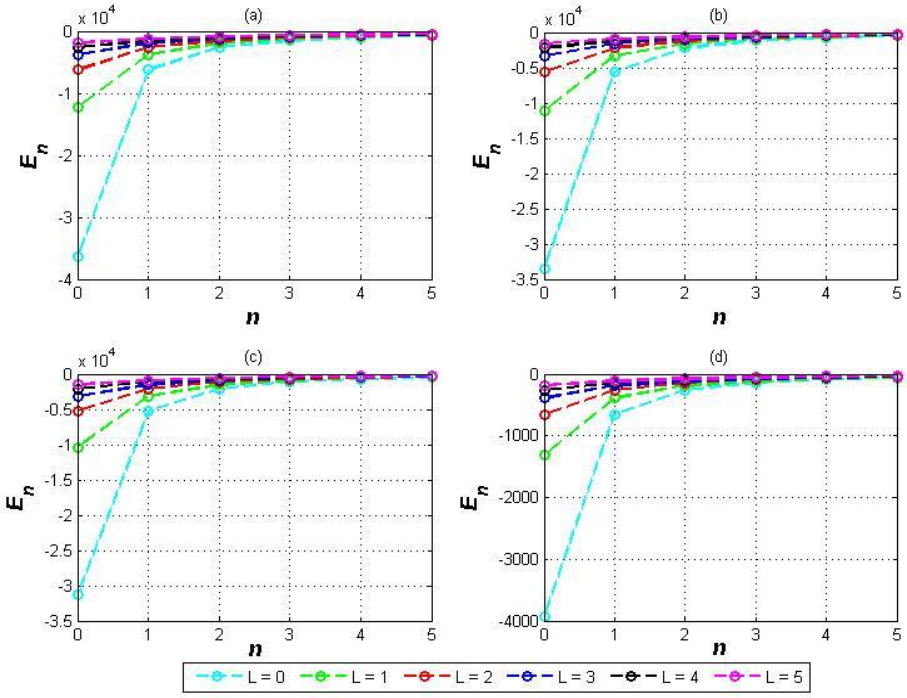

Figure 3. Spectrum Energy of q-deformed modified Eckart potential(E_n) in eV for the various n and L quantum numbers, and constant potential and minimal length parameter, γ=0.05 and α_{ML}=0.01 for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

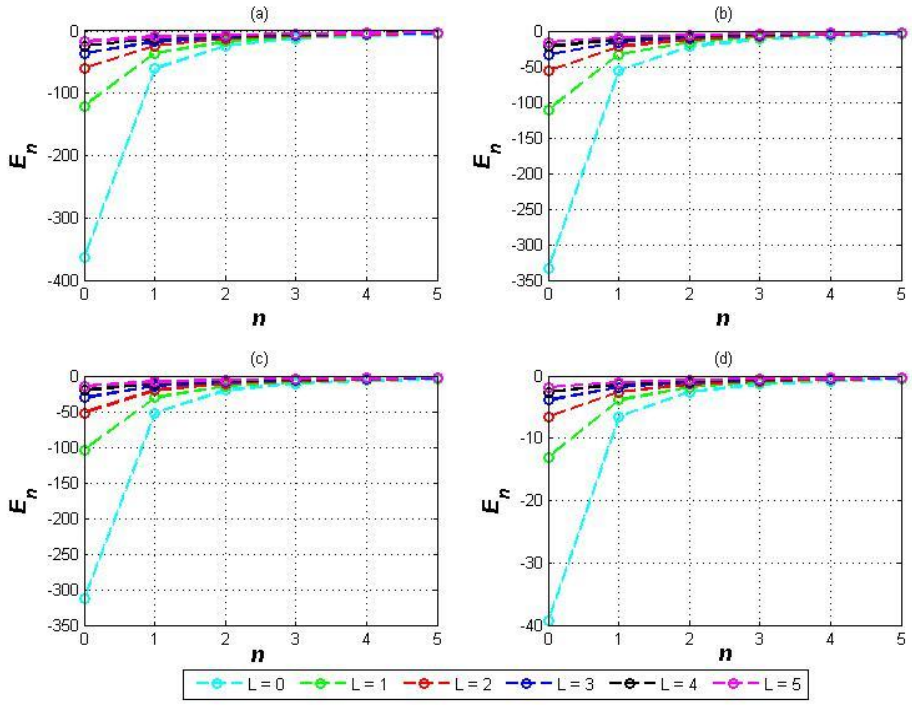

Figure 5. Spectrum Energy of q-deformed modified Eckart potential (E_n) in eV for the various *n* and *L* quantum numbers, and constant potential and minimal length parameter, γ=0.05 and α_{ML}=0.1 for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

By comparing Equations (26) and (45) we get the parameters of the hypergeometric differential equation expressed in Equation (26) for q-deformed modified Eckart potential as follows

$$\begin{aligned} &N=-1;\quad \mathrm{t}=\beta+\frac{1}{4};\ b=1; \\ &\mathrm{c}=\alpha-\frac{3}{4};\ W=\frac{k^2}{4}-(\alpha+\beta)^2; \\ &\delta=2\alpha+\frac{1}{2}=-\sigma+\frac{1}{2}; \\ &s=2\alpha+2\beta=-\sigma+\kappa \end{aligned} \qquad (52)$$

By using Equations (27-28) and (52) we obtain the wave function *f(z)* given as

$$f(z)=\begin{pmatrix} (-1)^n(1)^n C'\left(-\sigma+\frac{1}{2}\right)_n \\ {}_2F_1(-\mathrm{n},-\sigma+\kappa+\mathrm{n};-\sigma+\frac{1}{2};z) \end{pmatrix} \qquad (53)$$

Equation (53) is inserted into Equation (42) with $-\frac{\sigma}{2}=\alpha$, $\frac{\kappa}{2}=\beta$ then we obtain

$$U(z)=\begin{bmatrix} z^{-\frac{\sigma}{2}}(1-z)^{\frac{\kappa}{2}}(1)^n(-1)^n C'\left(-\sigma+\frac{1}{2}\right)_n \\ {}_2F_1(-\mathrm{n},-\sigma+\kappa+\mathrm{n};-\sigma+\frac{1}{2};z) \end{bmatrix} \qquad (54)$$

Since $\cosh^2(\gamma r) = z$, then we obtain the first three lowest un-normalized wave functions of the quantum system for q-deformed modified Eckart potential from Equation (54) as

$$U_o = C' \cosh^{-\sigma} \gamma r \left(-\sinh^2 \gamma r\right)^{\frac{\kappa}{2}} \tag{55}$$

$$U_1 = \begin{bmatrix} \left(-C' \cosh^{-\sigma} \gamma r \left(-\sinh^2 \gamma r\right)^{\frac{\kappa}{2}}\right) \\ \left(-\sigma + \frac{1}{2}\right) \\ \left(1 - \frac{(-\sigma + \kappa + 1)(\cosh \gamma r)^2}{\left(-\sigma + \frac{1}{2}\right)}\right) \end{bmatrix} \tag{56}$$

$$U_2 = \begin{bmatrix} \left(C' \cosh^{-\sigma} \gamma r \left(-\sinh^2 \gamma r\right)^{\frac{\kappa}{2}}\right) \\ \left(-\sigma + \frac{1}{2}\right)\left(-\sigma + \frac{3}{2}\right) \\ \left(1 - \frac{(2)(-\sigma + \kappa + 2)(\cosh \gamma r)^2}{\left(-\sigma + \frac{1}{2}\right)}\right) \\ + \frac{\left(\left(-\sigma + \kappa + 2\right)\right)}{\left(-\sigma + \frac{1}{2}\right)\left(-\sigma + \frac{3}{2}\right)} \end{bmatrix} \tag{57}$$

The graphs of the wave function for various *n, L*, *γ,* and α_{ML} are shown in Figures 2-6 are plotted using energy eigenvalues that are listed in Table 2-5 and un-normalized wave function equations expressed in Equations (55) – (57).

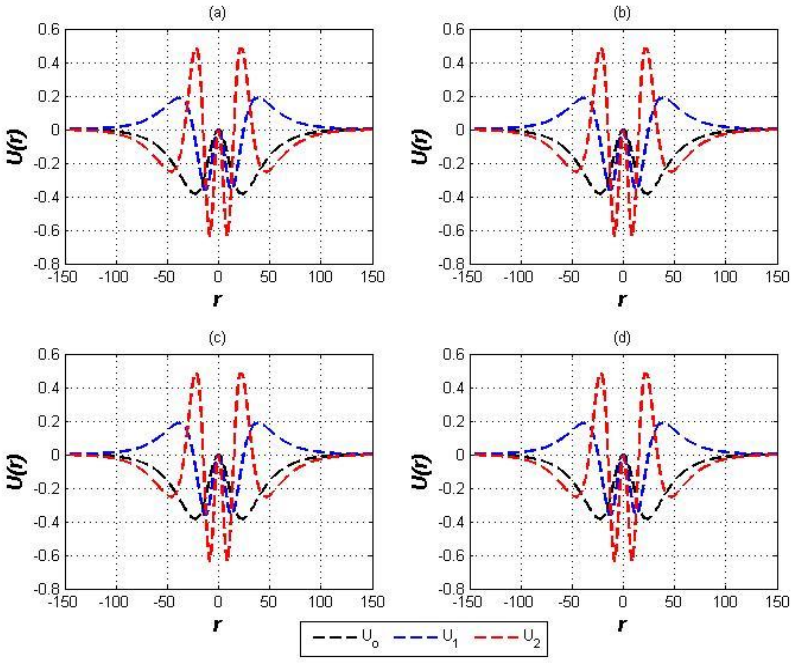

Figure 2. The graphs of the un-normalized wave function as a function of radial position, ground state (black line), the first excited (blue line) and second

excited (red line) wave functions for $\gamma = 0.05$, $\alpha_{ML} = 0.1$ and $L = 1$ for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

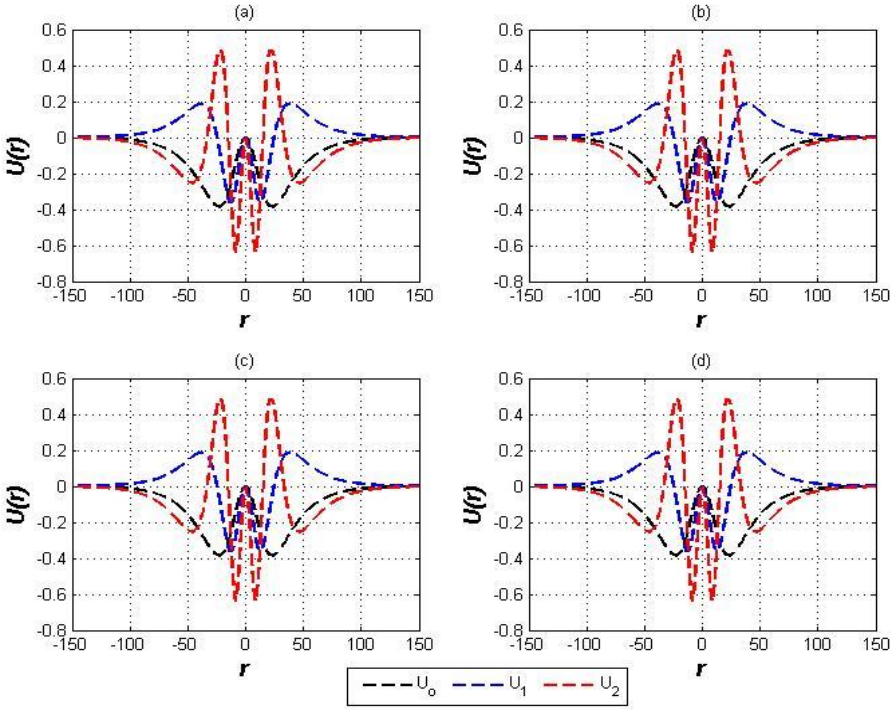

Figure 3. The graphs of the un-normalized wave function as a function of radial position, ground state (black line), the first excited (blue line), and second excited (red line) wave functions for $\gamma = 0.05$, $\alpha_{ML} = 0.01$ and $L = 1$ for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

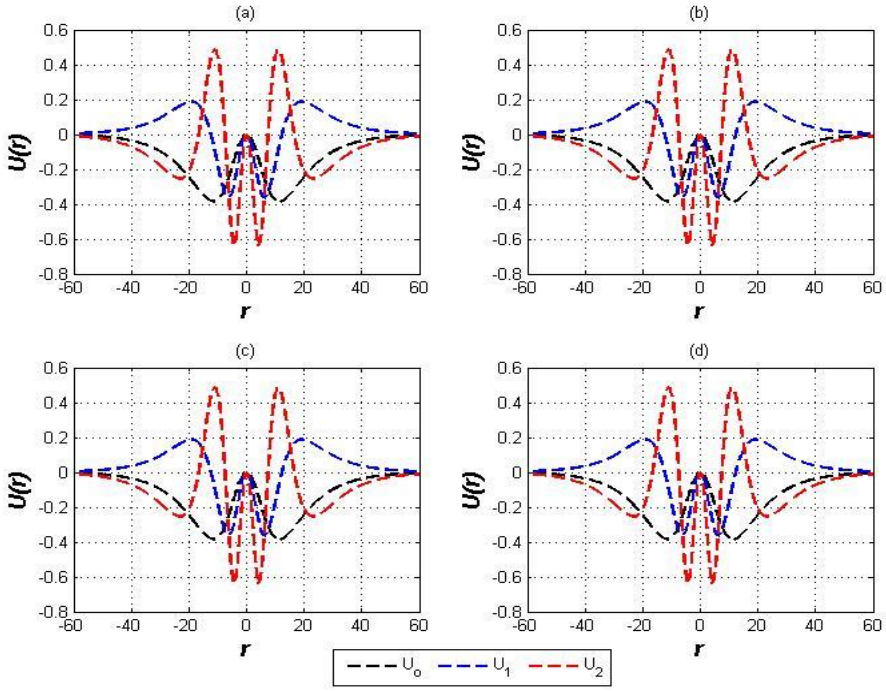

Figure 4. The graphs of the un-normalized wave function as a function of radial position, ground state (black line), the first excited (blue line), and second excited (red line) wave functions for $\gamma = 0.1$, $\alpha_{ML} = 0.1$ and $L = 1$ for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

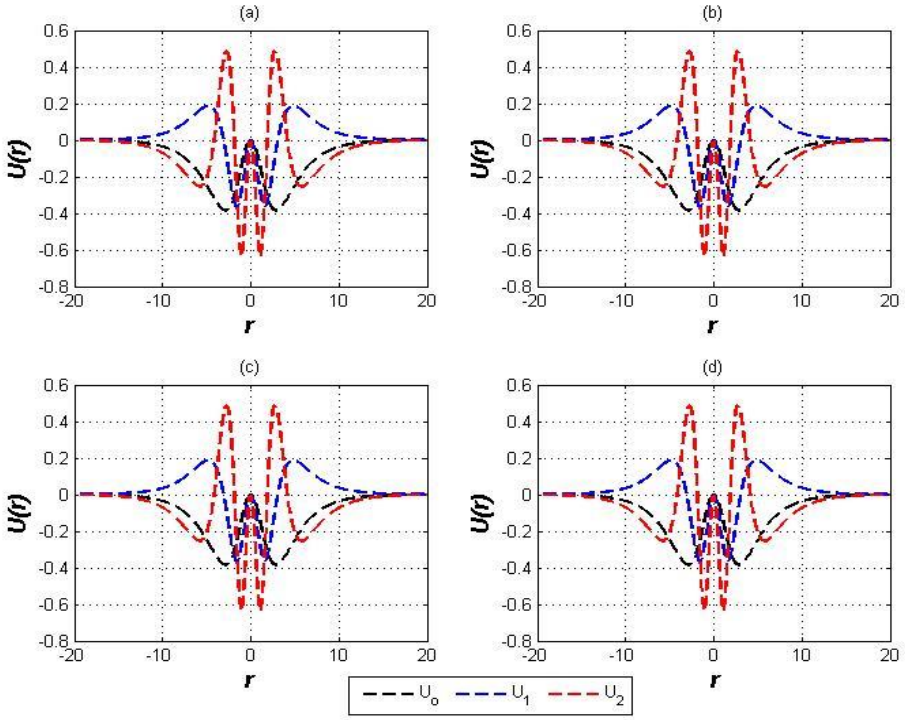

Figure 5. The graphs of the un-normalized wave function as a function of radial position, ground state (black line), the first excited (blue line), and second excited (red line) wave functions for $\gamma = 0.4$, $\alpha_{ML} = 0.01$ and $L = 1$ for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

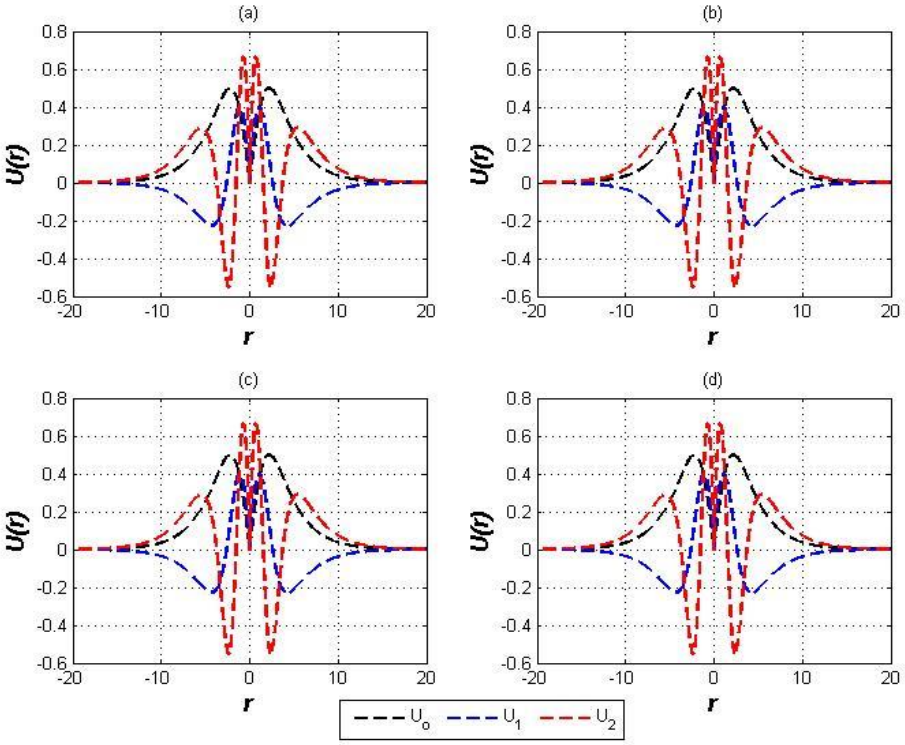

Figure 6. The graphs of the un-normalized wave function as a function of radial position, ground state (black line), the first excited (blue line), and second excited (red line) wave functions for $\gamma = 0.4$, $\alpha_{ML} = 0.01$, and $L = 0$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

The Rényi entropy is determined by using Equations (29) and (54) as follows

$$\xi = U^2(z) = \left(\begin{bmatrix} (C')^2 (1)^{2n} (-1)^{2n} \left(-\sigma + \frac{1}{2} \right)^2 \\ z^{-\sigma} (1-z)^{\kappa} \end{bmatrix} \right) \left(\begin{bmatrix} {}_2F_1 \begin{pmatrix} -\mathrm{n}, -\sigma + \kappa + \mathrm{n}; \\ -\sigma + \frac{1}{2}; z \end{pmatrix} \end{bmatrix}^2 \right) \quad (58)$$

$$\cosh^2(\gamma r) = z \rightarrow dr = \frac{dz}{2\gamma\sqrt{z(z-1)}} \quad (59a)$$

$$\begin{aligned} r = 0 &\rightarrow \cosh^2(\gamma r) = 1 \Rightarrow z = 1 \\ r = \infty &\rightarrow \cosh^2(\gamma r) = \infty \Rightarrow z = \infty \end{aligned} \quad (59b)$$

$$\begin{aligned} R_d(\xi) &= \frac{1}{1-d} \log 4\pi \int_0^{\infty} \xi(r)^d dr \\ &= \frac{1}{1-d} \left[\ln 4\pi + \ln \int_0^{\infty} \left\{ \xi(r) \right\}^d dr \right] \end{aligned} \quad (60)$$

Equation (58), (59a), and (59b) are inserted into Equation (60), then, by using simple algebraic manipulation from Equation (60) we have,

$$R_d(\xi) = \frac{1}{1-d} \left\{ \begin{array}{l} \ln 4\pi + \left(\dfrac{(-1)^{\kappa}}{2\gamma} (C')^2 \right) \ln(-i) \\ \left(\dfrac{}{\left(-\sigma + \dfrac{1}{2} \right)^2} \right) \\ \displaystyle\int_1^{\infty} \left[z^{-\sigma - 1/2} (1-z)^{\kappa - 1/2} \left[{}_2F_1 \begin{pmatrix} -\mathrm{n}, -\sigma + \kappa + \mathrm{n} \\ ; -\sigma + \frac{1}{2}; z \end{pmatrix} \right]^2 \right]^d dz \end{array} \right\} \quad (61)$$

By using integrals proposed by Bajpai (Bajpai, 1993), from equation (61) we get

$$R_d(\xi) = \left(\begin{array}{l} \dfrac{\ln 4\pi}{1-d} + \left(\dfrac{(-1)^{\kappa}}{2\gamma(1-d)} \right) \\ \left((C')^2 \left(-\sigma + \dfrac{1}{2} \right)^2 \right) \\ \left(\ln(-\mathrm{i}) \left\{ \dfrac{\left(\begin{array}{l} n!(-2n + \sigma - \kappa + 1) \\ {}_n\Gamma(-2n + \sigma - \kappa) \\ \Gamma(\kappa + \frac{1}{2} + n) \end{array} \right)}{(\sigma + \frac{1}{2} - n)_n \Gamma(\sigma + \frac{1}{2})} \right\}^d \right) \end{array} \right) \quad (62)$$

Equation (62) is the Rényi entropy of q-deformed modified Eckart Potential in the presence of minimal length. Equation (62) is rewritten as the square root of the quadratic absolute value of the Rényi entropy given as

$$\left|R_d(\xi)\right| = \left(\frac{1}{1-d} \left(\sqrt{ \left(\frac{\pi}{2} O \right)^2 + \left(\ln \left\{ 4\pi \frac{\left(\begin{matrix} n!(-2n+\sigma-\kappa+1) \\ {}_n\Gamma(-2n+\sigma-\kappa) \\ \Gamma(\kappa+\frac{1}{2}+n) \end{matrix} \right)^{dO}}{\left((\sigma+\frac{1}{2}-n)_n \Gamma(\sigma+\frac{1}{2}) \right)^{dO}} \right\} \right)^2 } \right) \right) \tag{63}$$

with $\left(\frac{(-1)^{\kappa}}{2\gamma} (C')^2 \left(-\sigma + \frac{1}{2} \right)^2 \right) = O$.

By using Equation (63) the Rényi entropy is visualized using Matlab as shown in Figures 7-9 for various *d, γ, α_{ML}, L, n,* and mass of the molecule.

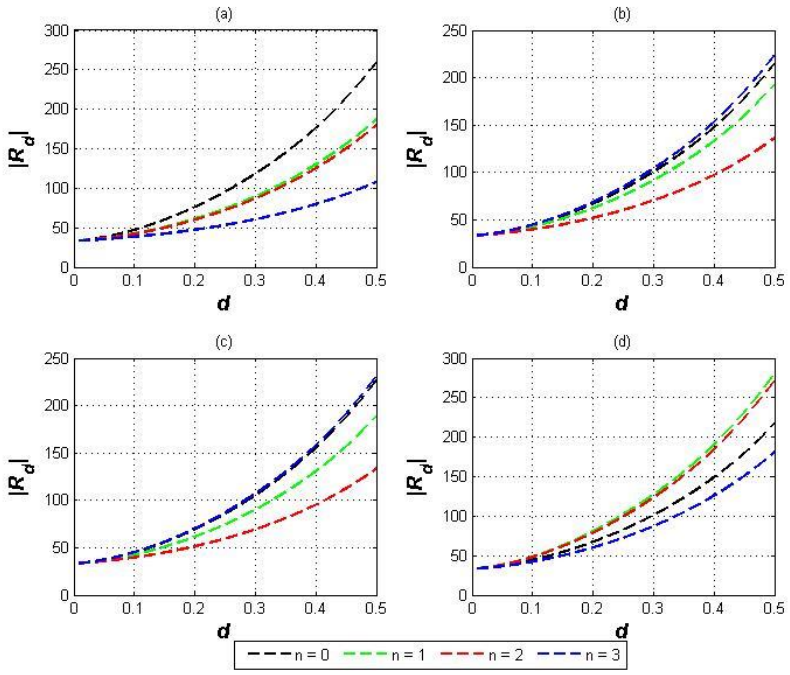

Figure 7. The graphs of the Rényi entropy for the various *n* and *d* for $\gamma = 0.05$, $\alpha_{ML} = 0.01$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

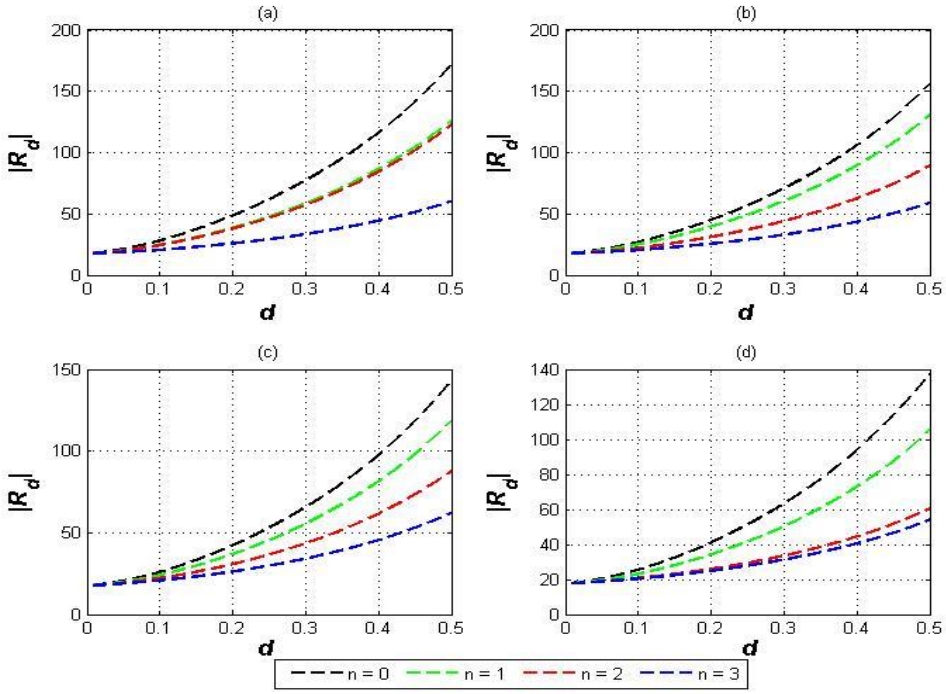

Figure 8. The graphs of the Rényi entropy for the various n and d for $\gamma = 0.1$, $\alpha_{ML} = 0.1$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

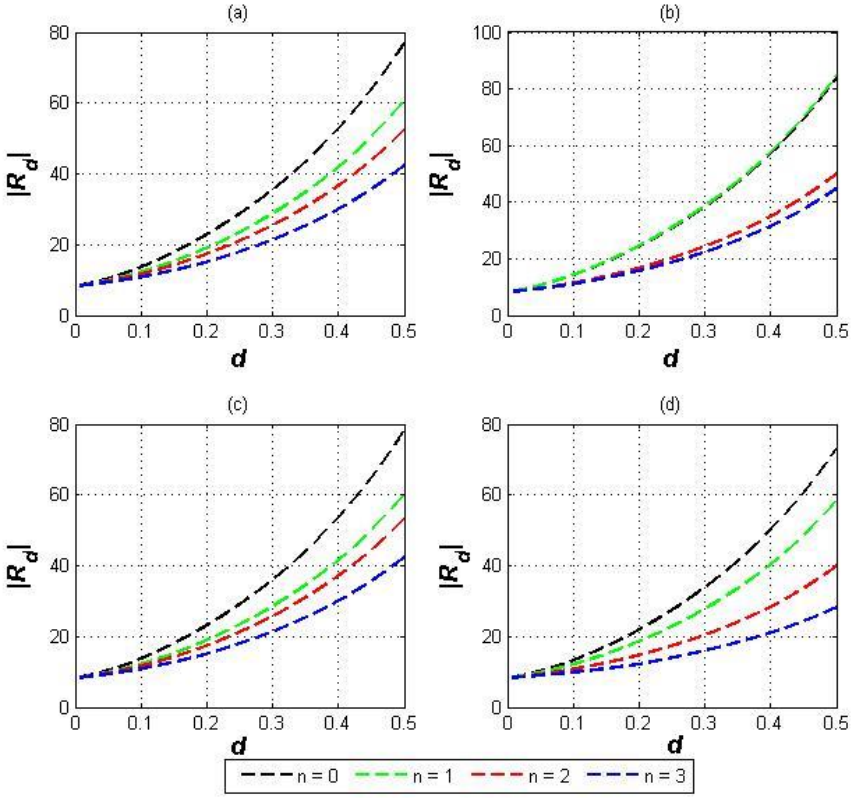

Figure 9. The graphs of the Rényi entropy for the various n and d for $\gamma = 0.4$, $\alpha_{ML} = 0.01$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules..

By inserting Equation (62) into Equation (31), we obtain the mass-energy parameter of the Schwarzschild black hole given as

$$M=\sqrt{\frac{4\pi\left(-iA\right)^{0}-1}{4\pi\left(1-d\right)}}$$

$$=\sqrt{\frac{4\pi\left(-i\left\{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)}\right)^{d}\right\}^{0}\right)-1}{4\pi\left(1-d\right)}} \quad (64)$$

Equation (64) is the mass-energy parameter of Schwarzschild black hole equation of q-deformed modified Eckart potential in minimal length formalism.
From Equation (64) we have

$$|M|=\sqrt{\frac{1}{4\pi\left(1-d\right)}\sqrt{16\pi^{2}\left\{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)}\right)^{2dO}\right\}+1}} \quad (65)$$

By using Equation (65) the mass-energy parameter of Schwarzschild black hole is visualized using Matlab as shown in Figures 10-12 for various d,γ,α_{ML},L,n and mass of the molecule.

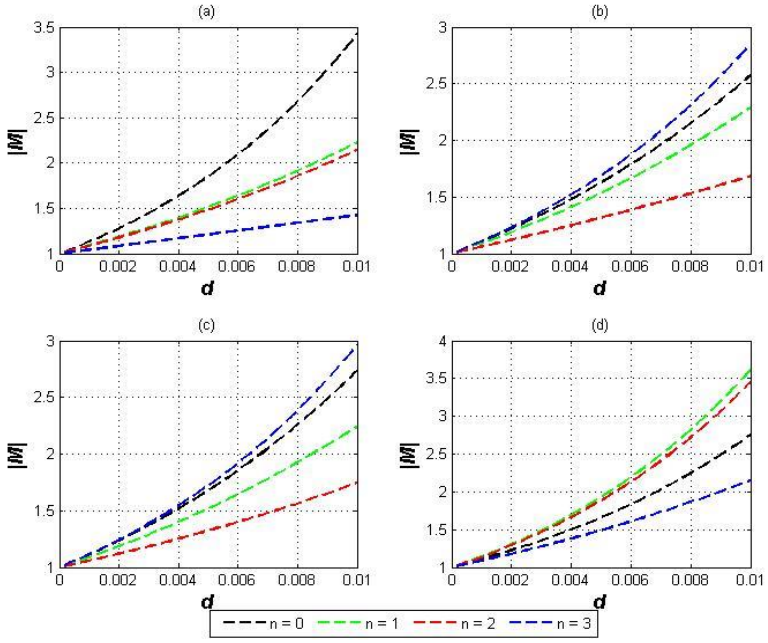

Figure 10. The graphs of mass-energy parameter of the Schwarzchild black hole for the various n and d for $\gamma=0.05$, $\alpha_{ML}=0.01$, and $L=1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

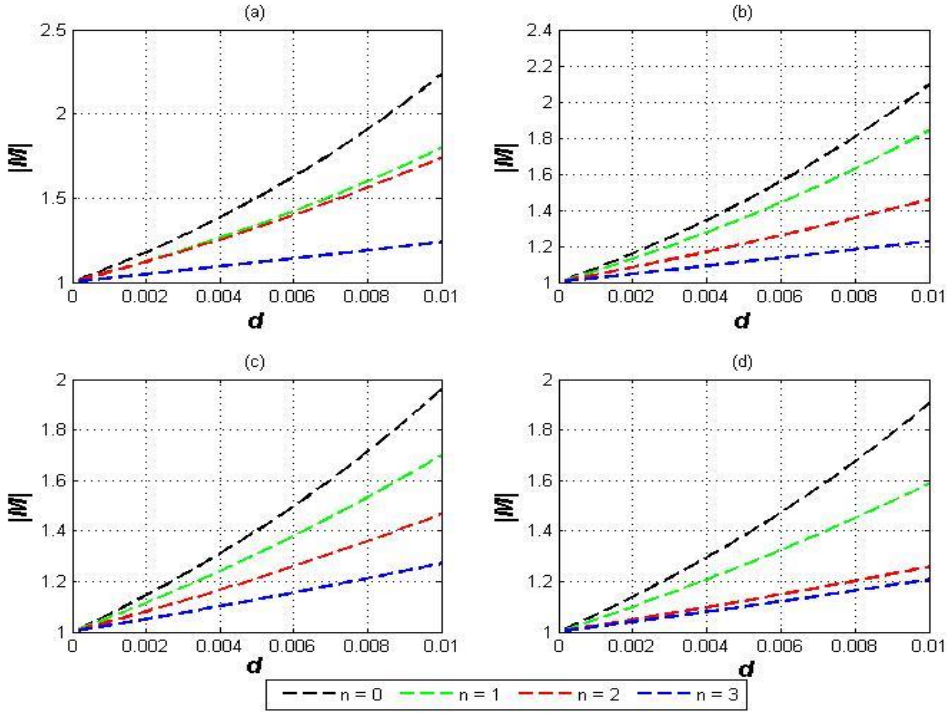

Figure 11. The graphs of mass-energy parameter of the Schwarzchild black hole for the various n and d for $\gamma = 0.1$, $\alpha_{ML} = 0.1$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules..

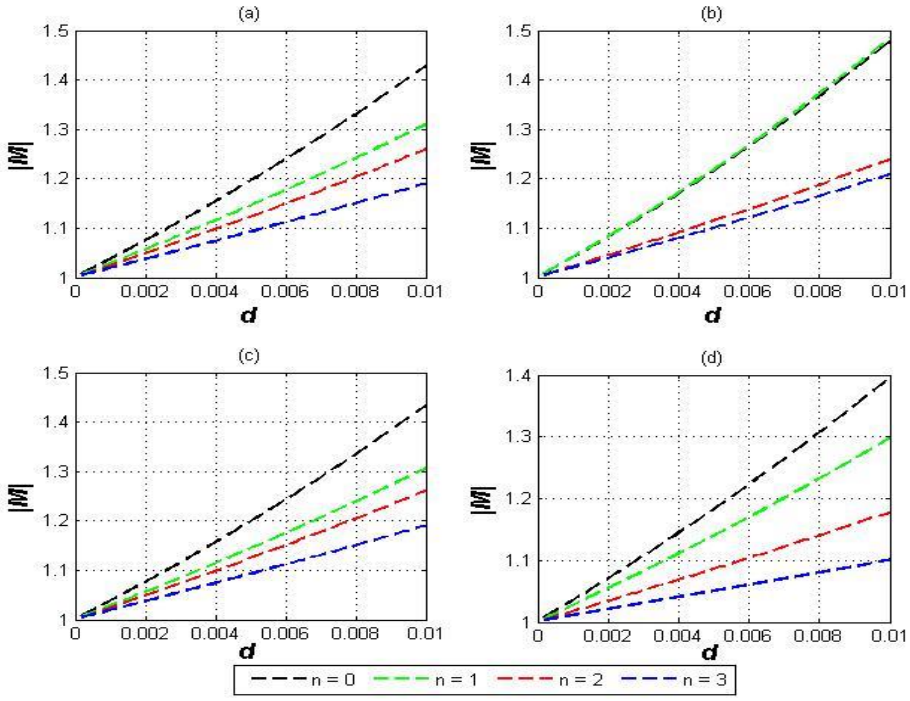

Figure 12. The graphs of mass-energy parameter of the Schwarzchild black hole for the various n and d for $\gamma = 0.4$, $\alpha_{ML} = 0.01$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

By inserting Equation (64) in Equation (32), then we get the temperature of Schwarzschild black hole equation for q-deformed modified Eckart potential given as

$$T_R = \frac{\left(-i\left\{\frac{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)}\right)}{(\sigma+\frac{1}{2}-n)_n\Gamma(\sigma+\frac{1}{2})}\right\}^d\right)^O}{2\sqrt{\frac{4\pi\left(-i\left\{\frac{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)}\right)}{(\sigma+\frac{1}{2}-n)_n\Gamma(\sigma+\frac{1}{2})}\right\}^d\right)^O - 1}{4\pi(1-d)}}} \quad (66)$$

Equation (66) is the temperature of Schwarzschild black hole equation of q-deformed modified Eckart potential in minimal length formalism.
Equation (66) can be rewritten as

$$|T_R| = \sqrt{\frac{\pi(1-d)A^{2O}}{\sqrt{16\pi^2A^{2O}+1}}}$$

$$= \sqrt{\frac{\pi(1-d)\left\{\frac{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)}\right)}{(\sigma+\frac{1}{2}-n)_n\Gamma(\sigma+\frac{1}{2})}\right\}^{2dO}}{\sqrt{16\pi^2\left\{\frac{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)}\right)}{(\sigma+\frac{1}{2}-n)_n\Gamma(\sigma+\frac{1}{2})}\right\}^{2dO}+1}}} \quad (67)$$

By using equation (67) the temperature of the Schwarzschild black hole is visualized using Matlab as shown in Figures 13-15 for various $d, \gamma, \alpha_{ML}, L, n$ and mass of the molecule.

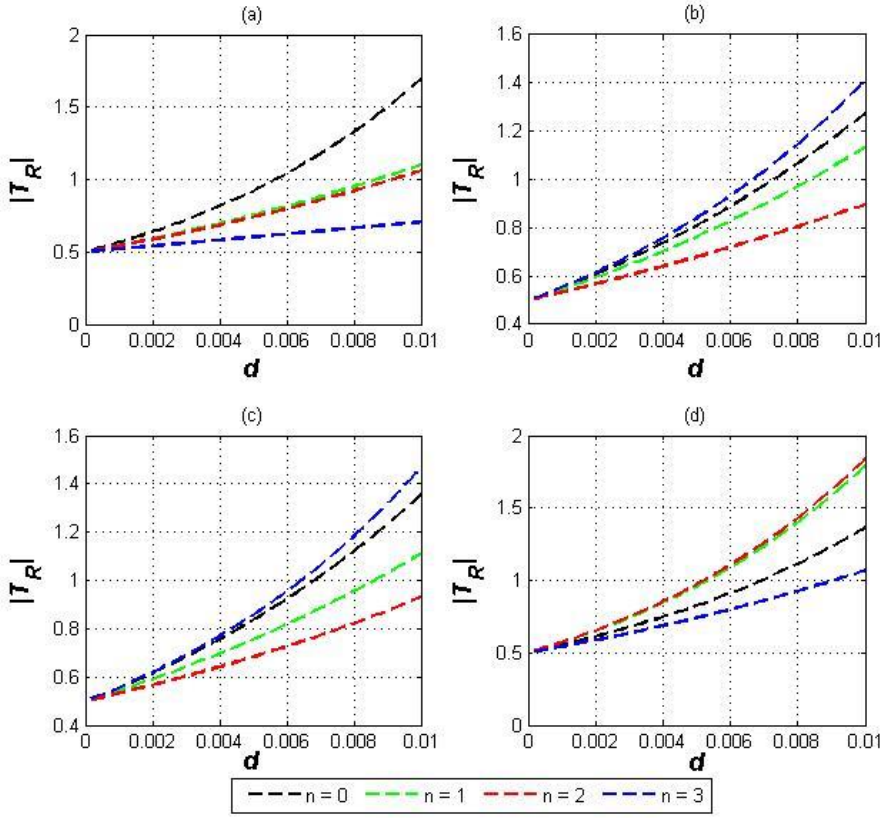

Figure 13. The graphs of the temperature of Schwarzschild black hole for the various n and d for $\gamma = 0.05$, $\alpha_{ML} = 0.01$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules..

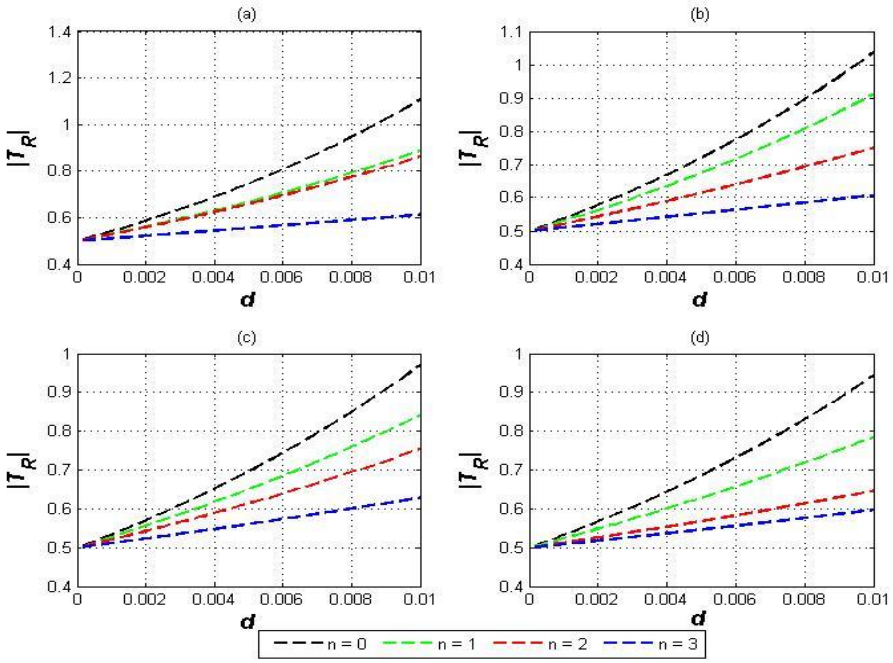

Figure 14. The graphs of the temperature of Schwarzschild black hole for the various n and d for $\gamma = 0.1$, $\alpha_{ML} = 0.1$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

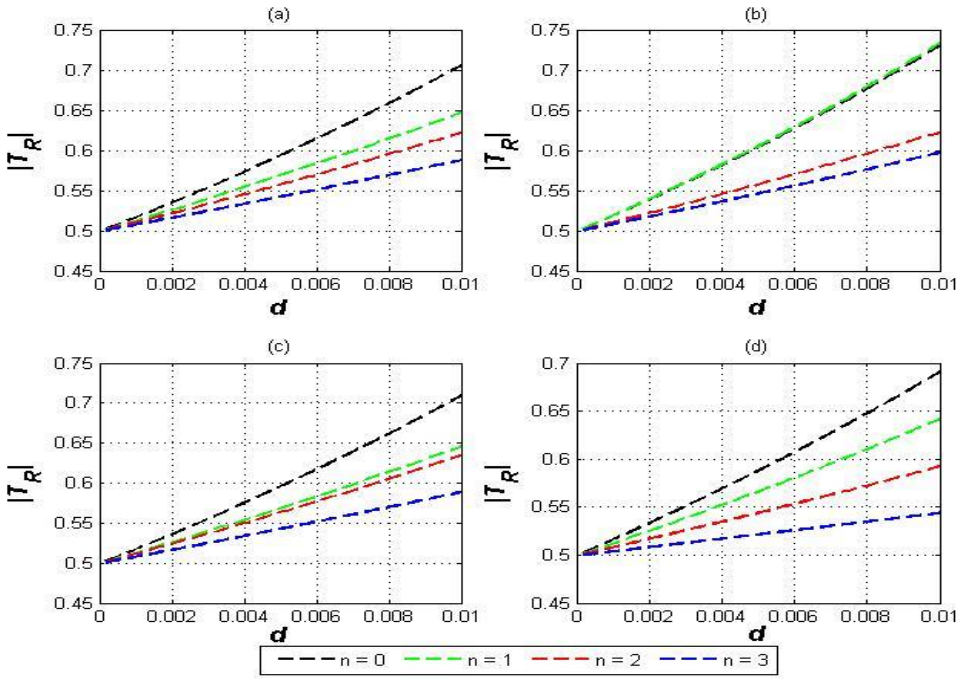

Figure 15. The graphs of the temperature of Schwarzschild black hole for the various n and d for $\gamma = 0.4$, $\alpha_{ML} = 0.01$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules..

By inserting Equation (64) into Equation (33), we get the heat capacity of Schwarzschild black hole equation for q-deformed modified Eckart potential is given as

$$C_R = \left(\frac{2}{(1-d)} \left(\frac{4\pi \left(-i \left\{ \frac{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)} \right)^d}{(\sigma+\frac{1}{2}-n)_n\Gamma(\sigma+\frac{1}{2})} \right\}^0 \right) - 1}{4\pi \left(-i \left\{ \frac{\left(\frac{n!(-2n+\sigma-\kappa+1)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)} \right)^d}{(\sigma+\frac{1}{2}-n)_n\Gamma(\sigma+\frac{1}{2})} \right\}^0 \right) - 2} \right) \right) \quad (68)$$

Equation (68) is the heat capacity of Schwarzschild black hole equation of q-deformed modified Eckart potential in minimal length formalism.

Equation (68) can be rewritten as

$$|C_R| = \left(\frac{1}{(1-d)} \sqrt{\frac{16\pi^2 \left\{ \frac{\left(n!(-2n+\sigma-\kappa+1) \right)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)} \right\}^{2dO} + 1}{4\pi^2 \left\{ \frac{\left(n!(-2n+\sigma-\kappa+1) \right)}{{}_n\Gamma(-2n+\sigma-\kappa)\Gamma(\kappa+\frac{1}{2}+n)} \right\}^{2dO} + 1}} \right) \quad (69)$$

By using Equation (69) the heat capacity of the Schwarzschild black hole is visualized using Matlab as shown in Figures 16-18 for various d, γ, α_{ML}, n, and mass of the molecule.

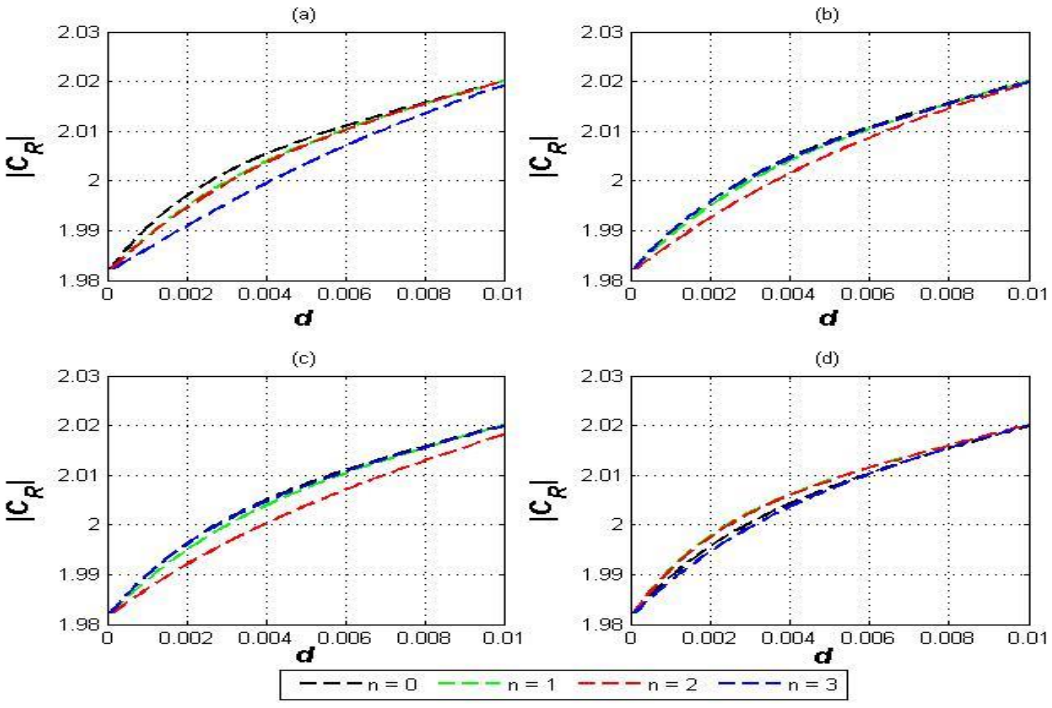

Figure 16. The graphs of the heat capacity of Schwarzschild black hole for the various n and d for $\gamma = 0.05$, $\alpha_{ML} = 0.01$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

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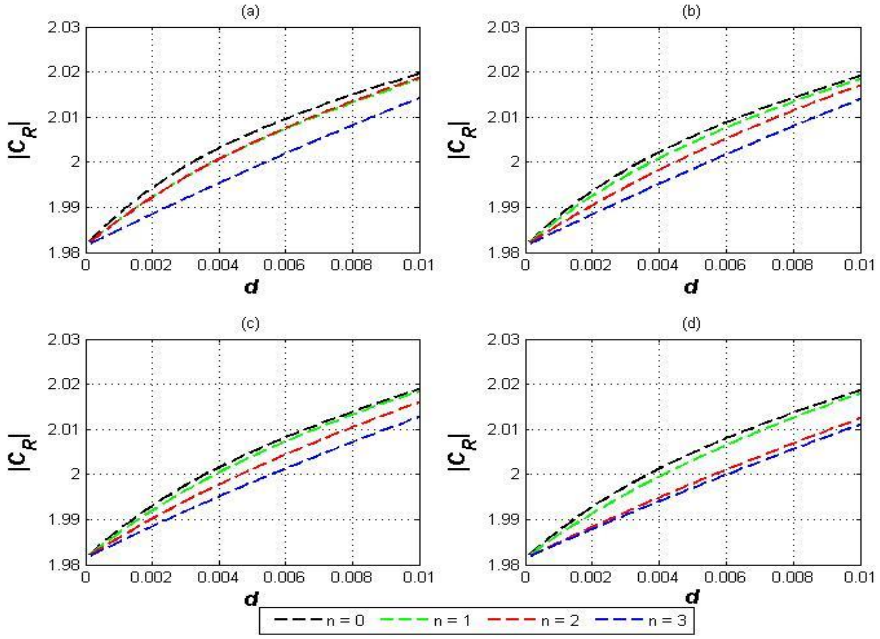

Figure 17. The graphs of the heat capacity of Schwarzschild black hole for the various n and d for $\gamma = 0.1$, $\alpha_{ML} = 0.1$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

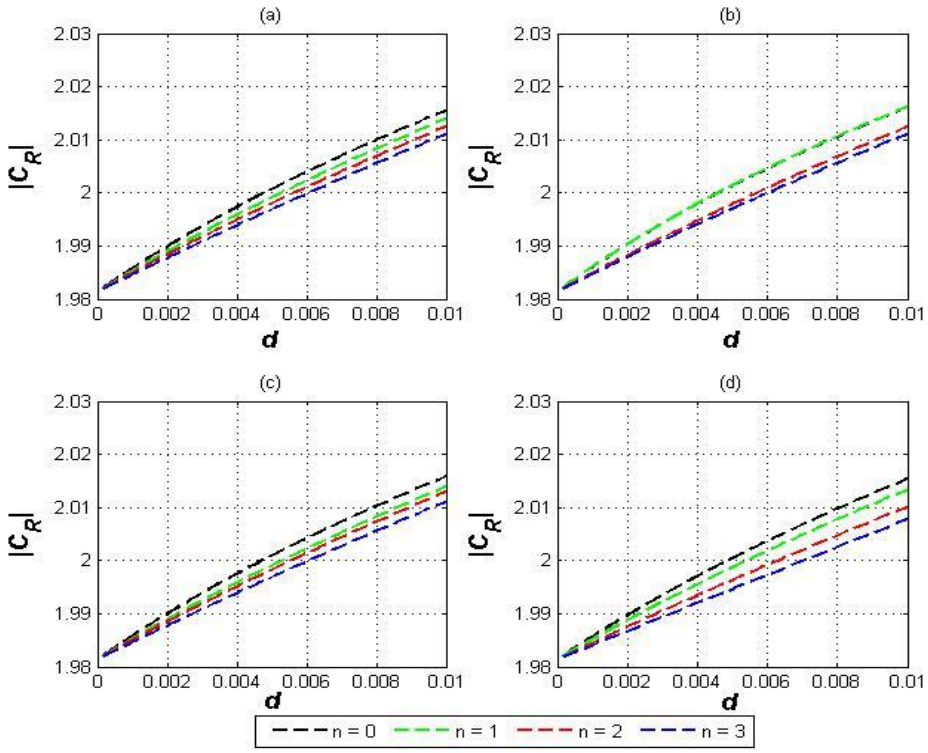

Figure 18. The graphs of the heat capacity of Schwarzschild black hole for the various n and d for $\gamma = 0.4$, $\alpha_{ML} = 0.01$, and $L = 1$, for (a) CO, (b) NO, (c) O_2, and (d) I_2 diatomic molecules.

The energy spectra equation which is expressed in equations (50-51) is a function of variables of radial and angular quantum number, potential width, minimal length, and molecules mass parameters, (*n, L, γ, α_{ML},* and *m*). All of these parameters gave the same effect to the calculated energy spectra. The increase of the value of the radial quantum *n* at constant values of (*L, γ, α_{ML},* and *m*) causes a decrease in the value of the energy spectra. The increase of the potential width *γ* at constant values of *n, L, α_{ML},* and *m* causes the decrease in the value of the energy spectra, and so for *L, α_{ML},* and *m*.

Figures (2-6) show that the radial quantum number *n*, and the potential width *γ*, have the most effect on the form, and the width or wavelength of the wave functions. The increase of the radial quantum number *n* causes the increase of the number of wave functions. By comparing figures (2) with (4), and figures (3) with (5), we could see that the wavelength of all of the wave functions is influenced by the values of the potential width *γ*. The wavelength of the wave function decreases by the increase of the value of the potential width *γ*, shown in Figures (2-5). As shown in Figures 2 and 3, there is no effect of the value change of minimal length parameter α_{ML} on the wave function, while the change of angular quantum numbers causes the wave function to be flat for r <-10 and r>10. It drops down as r approaches zero and has a symmetrical shape with r=0 as the axis of symmetry as shown in Figures 5 and 6.

Figures 7-9 show that for higher values of the potential width *γ*, the increase of radial quantum number *n* causes the decrease of the values of Rényi Entropy. It is also shown that the values of Rényi entropy decrease by the increase of the potential width γ.

Figures 10-12, show that the value of the mass-energy parameter of Schwarzschild black hole decrease by the increase of the value of the potential width, γ and the radial quantum number n. However, the value of the mass-energy parameter of the Schwarzschild black hole increase by the increase of the values of the entropy parameter d.

Figures 13-15 show that for higher values of potential width, γ the increase of radial quantum number n causes the decrease of the temperature, it is also shown that the temperature decreases by the increase of the potential width γ. The mass of the molecule gives a small effect on temperature.

From Figures 16-18 we could see that only the entropy parameter d that has the most effect on the specific heat capacity compared to other parameters, potential width, γ minimal length parameter α_{ML}, quantum numbers n,L, and mass of the molecule *m*.

CONCLUSION

The solution of energy eigenvalue and wave function equation of Schrödinger equation for q-deformed modified Eckart potential was analyzed in the presence of minimal length formalism using AIM. The Rényi entropy was obtained by using the radial wave function that was expressed in hypergeometric series. Then from the Rényi entropy, the mass-energy parameter, temperature, and the specific heat capacity of the quantum system were determined.

The result showed that the increase of the parameter of angular quantum numbers (*n* and *L*), potential width (γ) minimal length parameter (α_{ML}), and the reduced mass of the molecules (*m*) cause the decrease of the energy eigenvalues.

The radial quantum number (*n*), and the potential width (γ) have the most effect on the form and the wavelength of the wave functions. There was a very small effect of α_{ML} and the mass of the molecules.

For the Rényi entropy, the higher values of the potential width *γ* and the increase of radial quantum number *n* cause the decrease of its values. However, the values of the mass-energy parameter of Schwarzschild black hole increase by the increase of the values of entropy parameter *d*.

The results also showed that for higher values of potential width *γ* and the increase of radial quantum number *n* causes the decrease of the temperature, it is also shown that the temperature decreases by the increase of the potential width *γ*. The mass of the molecule gives a small effect on temperature. Only the entropy parameter *d* has the most effect on the specific heat capacity compared to other parameters.

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